

# Stability borders of feedback control of delayed measured systems

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When stabilization of unstable periodic orbits or fixed points by the method given by Ott, Grebogi, and Yorke (OGY) must be based on a measurement delayed by  $\tau$  orbit lengths, the performance of unmodified OGY method is expected to decline. For experimental considerations, it is desired to know the range of stability with minimal knowledge of the system. We find that unmodified OGY control fails beyond a maximal Lyapunov number of  $\lambda_{\max}=1+(1/\tau)$ . In this paper the area of stability is investigated both for OGY control of known fixed points and for difference control of unknown or inaccurately known fixed points. An estimated value of the control gain is given. Finally we outline what extensions must be considered if one wants to stabilize fixed points with Lyapunov numbers above  $\lambda_{\max}$ .

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## I. INTRODUCTION

The appearance of delay is a common problem in the control of chaotic systems. The effective delay time in any feedback loop is the sum of at least three delay times, the time of measurement, the time to compute the appropriate control amplitude, and the response time of the system to the applied control. If the applied control additionally must propagate through the system [1], these response times may extend to one or more cycle lengths.

In this paper we investigate time-discrete systems and focus on the question what limitations occur if one applies the control method given by Ott, Grebogi, and Yorke (OGY) [2] or difference feedback [3] in the presence of time delay.

Stabilization of chaotic systems by small perturbations in system variables [4] or control parameters [2] has become a widely discussed topic with applications in a broad area from technical to biological systems. The OGY method given by Ott, Grebogi, and Yorke [2] stabilizes unstable fixed points (or unstable periodic orbits utilizing a Poincaré surface section [5]) by feedback that is applied in the vicinity of the fixed point  $x^*$  of a discrete dynamics  $x_{t+1}=f(x_t, r)$ . The amplitude of the feedback  $r_t=r-r_0$  added to the control parameter  $r_0$  is proportional (with some user-adjustable parameter  $\varepsilon$  determining the strength of control) to the distance  $x-x^*$  from the fixed point,

$$r_t = \varepsilon(x_t - x^*), \quad (1)$$

and the feedback gain can be determined from a linearization around the fixed point: Neglecting higher order terms, we have

$$\begin{aligned} f(x_t, r_0 + r_t) &= f(x^*, r_0) + (x_t - x^*) \cdot \left( \frac{\partial f}{\partial x} \right)_{x^*, r_0} + r_t \cdot \left( \frac{\partial f}{\partial r} \right)_{x^*, r_0} \\ &= f(x^*, r_0) + \lambda(x_t - x^*) + \mu r_t \\ &= f(x^*, r_0) + (\lambda + \mu\varepsilon) \cdot (x_t - x^*), \end{aligned} \quad (2)$$

where the Taylor coefficients  $\lambda$  and  $\mu$  are fixed values for each orbit. The second expression vanishes for  $\varepsilon = -\lambda/\mu$ , that is, in linear approximation the system arrives exactly at the fixed point in the next time step,  $x_{t+1} = x^*$ .

As the uncontrolled system at hand is assumed to be unstable in the fixed point, we generally have the situation  $|\lambda| > 1$ . The system with applied control is stable, in linear approximation, if the absolute value of the eigenvalues of the iterated map is smaller than one,

$$|x_{t+1} - x^*| = |(\lambda + \mu\varepsilon) \cdot (x_t - x^*)| < |x_t - x^*|, \quad (3)$$

i.e.,  $|(\lambda + \mu\varepsilon)| < 1$ . Therefore  $\varepsilon$  must be chosen between  $(-1-\lambda)/\mu$  and  $(+1-\lambda)/\mu$ , and this interval is of width  $2/\mu$  and independent of  $\lambda$ . For OGY control the range in  $\lambda$  that can be controlled remarkably is not bounded, which will appear to be different for delayed measurement (see the next section).

It should be mentioned that the stability analysis of the one-dimensional case holds also for higher-dimensional systems provided there is only one unstable direction. One can transform on the eigensystem of the Jacobi matrix  $\partial f/\partial r$  and finds again the equations of the one-dimensional case, reflecting that one only needs to apply control in the unstable direction (see, e.g., Refs. [6,7]).

## II. DELAYED CONTROL OF ITERATED MAPS

We want to know what limitations occur if the OGY rule is applied without modification. In OGY control, the control parameter  $r_t$  is time dependent, and without loss of generality we assume that  $x^*=0$  and that  $r_t=0$  if no control is applied. For simplicity, we discuss the case  $\tau=1$  first. For one time step delay, instead of  $r_t = \varepsilon x_t$  we have the proportional feedback rule

$$r_t = \varepsilon x_{t-1}. \quad (4)$$

Using the time-delayed coordinates  $(x_t, x_{t-1})$ , the linearized dynamics of the system with applied control is given by

$$\begin{pmatrix} x_{t+1} \\ x_t \end{pmatrix} = \begin{pmatrix} \lambda & \mu\varepsilon \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ x_{t-1} \end{pmatrix}. \quad (5)$$

The eigenvalues of  $\begin{pmatrix} \lambda & \mu\varepsilon \\ 1 & 0 \end{pmatrix}$  are given by  $\alpha_{1,2} = (\lambda/2) \pm \sqrt{(\lambda^2/4) + \varepsilon\mu}$ . Hence control can be achieved with  $\varepsilon$  being in an interval  $]-1/\mu, (1-\lambda)/\mu[$  with the width  $(2-\lambda)/\mu$ .

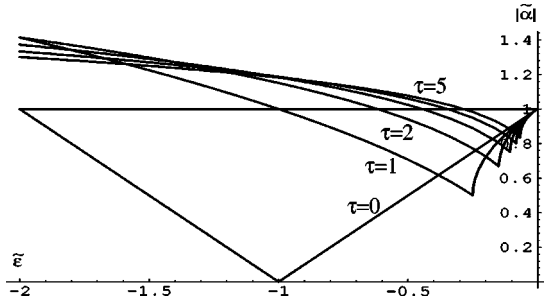


FIG. 1. Control intervals for several time delays  $\tau=0, \dots, 5$ : The plots show the maximal absolute value of the eigenvalues as a function of the rescaled control gain  $\tilde{\varepsilon}$ . Values of  $|\tilde{\alpha}|=1/\lambda$  correspond to  $|\alpha|=1$  in (6) without rescaling, so one can obtain the range  $]\varepsilon_-, \varepsilon_+[$  for which control is successfully achieved.

In contrast to the not-delayed case, we have a requirement  $\lambda < 2$  for the Lyapunov number, i.e., the direct application of the OGY method fails for systems with a Lyapunov number of 2 and higher. This limitation is caused by the additional degree of freedom introduced in the system due to the time delay.

Now we consider the general case. If the system measurement is delayed by  $\tau$  steps,  $r_t = \varepsilon x_{t-\tau}$  we write the dynamics in delayed coordinates  $(x_t, x_{t-1}, x_{t-2}, \dots, x_{t-\tau})^T$ ,

$$\begin{pmatrix} x_{t+1} \\ \vdots \\ \vdots \\ x_{t-\tau+1} \end{pmatrix} = \begin{pmatrix} \lambda & 0 & \cdots & \cdots & 0 & \varepsilon\mu \\ 1 & 0 & & & & 0 \\ 0 & 1 & \ddots & & & \vdots \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ \vdots \\ \vdots \\ x_{t-\tau} \end{pmatrix}.$$

The characteristic polynomial is given by (we define rescaled coordinates  $\tilde{\alpha} := \alpha/\lambda$  and  $\tilde{\varepsilon} = \varepsilon\mu/\lambda^{\tau+1}$ )

$$\begin{aligned} 0 &= P(\alpha) = (\alpha - \lambda)\alpha^\tau - \varepsilon\mu, \\ 0 &= P(\tilde{\alpha}) = (\tilde{\alpha} - 1)\tilde{\alpha}^\tau - \tilde{\varepsilon}. \end{aligned} \tag{6}$$

Figure 1 shows the maximum of the absolute value of the eigenvalues, for  $\tau=0, 1, \dots, 5$ . In rescaled coordinates  $\tilde{\alpha} = 1/\lambda$  corresponds to a control interval  $]\tilde{\varepsilon}_\pm(\tau, \lambda)[$ . For

$$\lambda_{\max} = 1 + \frac{1}{\tau} \tag{7}$$

the control interval vanishes, and for  $\lambda \geq \lambda_{\max}(\tau)$  no control is possible. Equation (7) and the subsequently derived stability diagrams are the main result of this paper and are transferred to difference control in Sec. IV.

If we look at the Lyapunov exponent  $\Lambda := \ln \lambda$  instead of the Lyapunov number, we find with  $\ln x < (x-1)$  the inequality

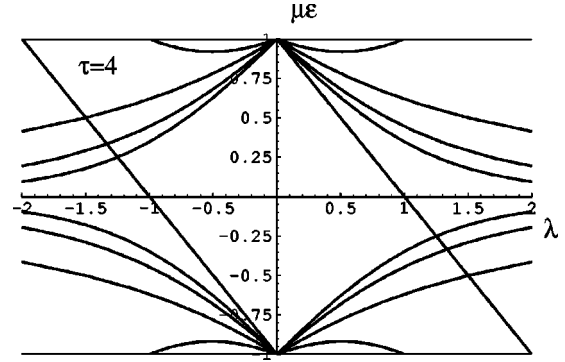


FIG. 2. Complete Jury diagram for  $\tau=4$  (see Appendix B).

$$\Lambda_{\max} \cdot \tau < 1. \tag{8}$$

Therefore, delay time and Lyapunov exponent limit each other if the system is to be controlled. This is consistent with the loss of knowledge in the system by exponential separation of trajectories.

### III. STABILITY DIAGRAMS BY THE JURY CRITERION

For small  $\tau$  one can derive easily the borders of the stability area with help of the Jury criterion [8] (see Appendix A). For  $\tau=1$ , the Jury coefficients are given by  $\alpha_1 = -\lambda/(1 + \mu\varepsilon)$  and  $\alpha_2 = -\mu\varepsilon$ , and for  $\tau=2$  to  $\tau=4$  the corresponding expressions are shown in Appendix B.

The equations  $\alpha_i = \pm 1$  can, although the characteristic polynomial (6) itself is of degree 5, be solved for one variable (giving large expressions). The complete set of lines is shown in Fig. 2 for  $\tau=4$  and illustrates the redundancy of the inequalities generated by the Jury table. Only four (three for  $\tau=1$ ) of the  $2n$  inequalities constitute the border of stability, and unfortunately it seems one must select them by hand. Control is only necessary for  $|\lambda| > 1$ , and folding the relevant

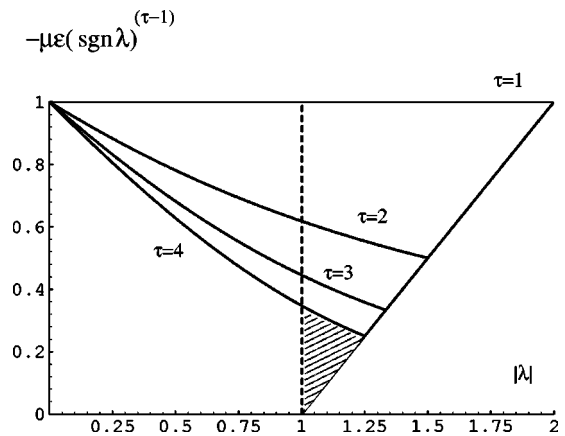


FIG. 3. Stability areas for  $\tau=1, 2, 3, 4$ , combined. Only for  $|\lambda| > 1$  control is necessary (dashed line), and the stability area (shaded for  $\tau=4$ ) extends to  $|\lambda_{\max}| = 2, 3/2, 4/3, 5/4$ . Note that still both positive and negative  $\lambda$  can be controlled. The abscissa  $-\mu\varepsilon(\text{sgn } \lambda)^{(\tau-1)}$  takes into account that for odd  $\tau$  a negative  $\mu\varepsilon$  is required, independent of the sign of  $\lambda$ .

stability area into the same quadrant gives Fig. 3 showing how  $\lambda_{\max}$  decreases for increasing  $\tau$ .

#### IV. STABILIZING UNKNOWN FIXED POINTS

As the OGY approach discussed above requires the knowledge of the position of the fixed point, one may wish to stabilize purely by feeding back differences of the system variable at different times. This becomes relevant in the case of parameter drifts [9] which often can occur in experimental situations. A time-continuous strategy  $r(t) = \varepsilon[x(t) - x(t - \tau_d)]$  has been introduced by Pyragas [10], where  $r(t)$  is updated continuously and  $\tau_d$  matches the period of the unstable periodic orbit. The time-discrete counterpart (i.e., control amplitudes are calculated every Poincaré section) is defined by the difference control scheme [3]. For control without delay, a simple difference control strategy [3,7,11] is possible for  $\varepsilon\mu = -\lambda/3$ , and eigenvalues of modulus smaller than unity of the matrix  $\begin{pmatrix} \lambda + \varepsilon\mu & -\varepsilon\mu \\ 1 & 0 \end{pmatrix}$  are obtained only for  $-3 < \lambda < +1$ , so this method stabilizes only for oscillatory repulsive fixed points with  $-3 < \lambda < -1$  [3,7,11].

We can proceed in a similar fashion as for OGY control. In the presence of  $\tau$  steps with delay, the linearized dynamics of a simple difference feedback  $r_t = \varepsilon(x_{t-\tau} - x_{t-\tau-1})$  are given by

$$\begin{pmatrix} x_{t+1} \\ \vdots \\ \vdots \\ x_{t-\tau} \end{pmatrix} = \begin{pmatrix} \lambda & 0 & \cdots & 0 & \varepsilon\mu & -\varepsilon\mu \\ 1 & 0 & & & & 0 \\ 0 & 1 & \ddots & & & \vdots \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ \vdots \\ \vdots \\ x_{t-\tau-1} \end{pmatrix}$$

in delayed coordinates  $(x_t, x_{t-1}, \dots, x_{t-\tau-1})$ , and the characteristic polynomial is given by

$$0 = (\alpha - \lambda)\alpha^{\tau+1} + (1 - \alpha)\varepsilon\mu. \quad (9)$$

As we must use  $x_{t-\tau-1}$  in addition to  $x_{t-\tau}$ , the system is of dimension  $\tau+2$ , and the lower bound of Lyapunov numbers that can be controlled are found to be

$$\lambda_{\text{inf}} = -\frac{3+2\tau}{1+2\tau} = -\left(1 + \frac{1}{\tau+1/2}\right) \quad (10)$$

and the asymptotic control amplitude at this point is

$$\varepsilon\mu = \frac{(-1)^\tau}{1+2\tau}. \quad (11)$$

The stability area in the  $(\mu\varepsilon, \lambda)$  plane is bounded by the lines  $\alpha_i = \pm 1$  where  $\alpha_i$  are the coefficients given by the Jury criterion [8] (see Appendix A). For  $\tau=0$ , the Jury coefficients are  $\alpha_1 = (-\lambda + \varepsilon\mu)/(1 + \varepsilon\mu)$  and  $\alpha_2 = \varepsilon\mu$ . For  $\tau=1$  to  $\tau=3$ , the Jury coefficients are shown in Appendix C. (See Fig 4.)

The controllable range is smaller than for the unmodified OGY method, and is restricted to oscillatory repulsive fixed

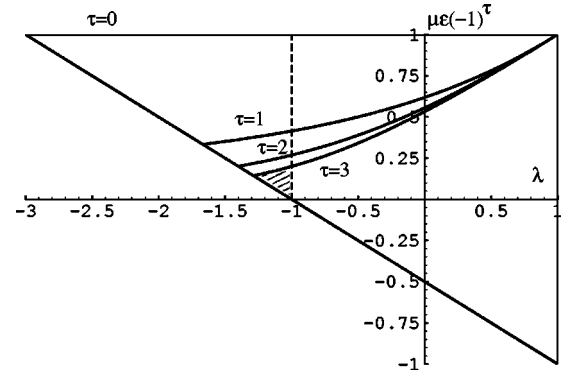


FIG. 4. Difference feedback for  $\tau=0,1,2,3$ : Stability borders derived by the Jury criterion (see Appendix C). The stability diagram of the nondelayed case  $\tau=0$  has already been given in Ref. [3]. From  $\lambda=-1$  (dashed line) to  $\lambda=+1$  the system is stable without control. For each  $\tau$ , control is effective only within the respective area (shaded for  $\tau=3$ ).

points with  $\lambda_{\text{inf}} < \lambda \leq -1$ . A striking observation is that inserting  $\tau + \frac{1}{2}$  for  $\tau$  in Eq. (7) exactly leads to the expression in Eq. (10) which reflects the fact that the difference feedback control can be interpreted as a discrete first derivative, taken at time  $t - (\tau + \frac{1}{2})$ . Thus the controllability relation (8) holds again.

As  $\lambda^{-1}$  is implying a natural time scale (that of exponential separation) of an orbit, it is quite natural that control becomes limited by a border proportional to a product of  $\lambda$  and a feedback delay time. Already without the additional difficulty of a measurement delay this is expected to appear for any control scheme that itself is using time-delayed feedback: For example, the extensions of time-discrete control schemes discussed in Ref. [12] with an inherent Lyapunov number limitation due to memory terms, and the experimentally widely applied time-continuous schemes Pyragas and ET DAS [13–15]. Here Pyragas control has the Lyapunov exponent limitation  $\Lambda \tau_p \leq 2$  together with the requirement of the Floquet multiplier of the uncontrolled orbit having an imaginary part of  $\pi$ , meaning that deviations from the orbit after one period experience are flipped around the orbit by that angle, which is quite the generic case [16]. This nicely corresponds with the requirement of a negative Lyapunov number that appears in difference control. A positive Lyapunov number in the time-discrete picture corresponds to a zero flip of the time-continuous orbit, and is consistently uncontrollable in both schemes.

Recently, the influence of a control loop latency has also been studied for continuous time-delayed feedback [16] by Floquet analysis, obtaining a critical value for the measurement delay  $\tau$ , that corresponds to a maximal Lyapunov exponent  $\log|\lambda_{\text{inf}}| = \Lambda \tau_p = 1/1/2 + \tau/\tau_p$ , where  $\tau_p$  is the orbit length and matched feedback delay. By the log inequality that again translates (for small Lyapunov exponents) to our result for the time-discrete difference control. An exact coincidence could not be expected, as in Pyragas control the feedback difference is computed continuously sliding with the motion along the orbit, where in difference control it is evaluated within each Poincaré section. For the ET DAS scheme with latency, a detailed analysis is performed in Ref.

[17], showing that the range of stability can be extended compared to the Pyragas scheme. Although the time-continuous case (as an *a priori* infinite-dimensional delay-differential system) could exhibit much more complex behavior, it is, however astonishing that for all three methods, OGY, difference, and Pyragas control, the influence of measurement delay mainly results in the same limitation of the controllable Lyapunov number.

**V. CONCLUSIONS**

Delayed measurement is a generic problem that can appear in controlling chaos experiments. In some situations it may be technically impossible to extend the control method, then one wants to know the stability borders with minimal knowledge of the system.

We have shown that both OGY control and difference control cannot control orbits with an arbitrary Lyapunov number if there is only delayed knowledge of the system. The maximal Lyapunov number up to which an unstable orbit can be controlled is given by  $1+(1/\tau)$  for OGY control and  $1+1/(\tau+1/2)$  for difference control. For small  $\tau$  the stability borders can be derived by the Jury criterion, so that the range of values for the control gain  $\varepsilon$  can be determined from the knowledge of the Taylor coefficients  $\lambda$  and  $\mu$ . If one wants to overcome these limitations, one must modify the control strategy.

This can be done either by applying control rhythmically [3,6,7,9] being equivalent in a formal sense to controlling the  $\tau+1$ -fold iterate (for OGY control), respectively,  $\tau+2$ -fold iterate (for difference control) of the original system [6,7]. However, for larger values of  $\tau$ , the required values for the control gain grow exponentially with  $\tau$ , because the possibility of applying control in the intermediate time steps is not used.

The other possibility to improve control are memory methods [6,7,9,12]. For negative  $\lambda$ , in the nondelayed case  $\tau=0$  the stability area can be extended by two methods including an averaged, respectively, decaying memory and requiring only one extra parameter [12].

For arbitrary  $\lambda$  and delayed measurement, an improved control even ensuring all eigenvalues to become zero can be

achieved by memory methods that take into account control amplitudes applied in previous time steps [6,7].

**APPENDIX A: THE JURY CRITERION**

The Jury criterion [8] gives a sufficient and necessary condition that all roots of a given polynomial are of modulus smaller than unity. Given a polynomial  $P(x)=a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0$ , one applies the iterative scheme of the Jury table:

$$\forall_{0 \leq i \leq n} \quad b_i := a_{n-i}$$

$$\alpha_n := b_n/a_n$$

$$\forall_{1 \leq i \leq n} \quad a_{i-1}^{\text{new}} := a_i - \alpha_n b_i$$

giving  $\alpha_n$  and coefficients  $a_{n-1} \dots a_0$  for the next iteration.

The Jury criterion states that the eigenvalues are of modulus smaller than unity if and only if  $\forall_{1 \leq i \leq n} |\alpha_i| < 1$ . The criterion gives  $2n$  (usually partly redundant) inequalities that define hypersurfaces in coefficient space. These hypersurfaces are given by algebraic equations; it is not necessary to compute the roots of the polynomial.

Whereas the Jury criterion is extremely helpful for small  $n$  and for numeric purposes, the hypersurface equations become very complex for large  $n$ , and one must select the relevant hypersurface equations. Two additional necessary conditions (“check-first conditions”) for stability are  $(-1)^n a_n P(-1) > 0$  and  $a_n P(+1) > 0$ .

**APPENDIX B: JURY COEFFICIENTS FOR UNMODIFIED OGY CONTROL**

For  $\tau=2$ , the Jury coefficients take the values

$$\alpha_3 = -\mu\varepsilon, \quad \alpha_2 = -\frac{\lambda\mu\varepsilon}{1-(\mu\varepsilon)^2}, \quad \alpha_1 = -\frac{\lambda}{1-(\mu\varepsilon)^2 + \lambda\mu\varepsilon}$$

and among the crossing points of the six lines given by  $\alpha_i = \pm 1$  one finds the maximal Lyapunov number  $\lambda = \pm 3/2$ . For  $\tau=3$  the Jury coefficients are

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$$\alpha_4 = -\mu\varepsilon, \quad \alpha_3 = -\frac{\lambda\mu\varepsilon}{1-(\mu\varepsilon)^2}, \quad \alpha_2 = \frac{\lambda^2\mu\varepsilon}{-1+2(\mu\varepsilon)^2+\lambda^2(\mu\varepsilon)^2-(\mu\varepsilon)^4}, \quad \alpha_1 = \frac{\lambda-\lambda(\mu\varepsilon)}{-1+\mu\varepsilon+\lambda^2\mu\varepsilon+(\mu\varepsilon)^2-(\mu\varepsilon)^3}.$$

For  $\tau=4$ , the borders given by the Jury coefficients are already described by algebraic equations of higher order,

$$\alpha_5 = -\mu\varepsilon, \quad \alpha_4 = \frac{\lambda\mu\varepsilon}{-1+(\mu\varepsilon)^2}, \quad \alpha_1 = \frac{\lambda(1-\lambda\mu\varepsilon-(\mu\varepsilon)^2)}{-1+\lambda\mu\varepsilon+\lambda^3\mu\varepsilon+2(\mu\varepsilon)^2+\lambda^2(\mu\varepsilon)^2-\lambda(\mu\varepsilon)^3-(\mu\varepsilon)^4},$$

$$\alpha_3 = \frac{\lambda^2\mu\varepsilon}{-1+2\mu\varepsilon^2+\lambda^2\mu\varepsilon^2-(\mu\varepsilon)^4}, \quad \alpha_2 = \frac{\lambda^3\mu\varepsilon}{-1+3(\mu\varepsilon)^2+2\lambda^2(\mu\varepsilon)^2+\lambda^4(\mu\varepsilon)^2-3(\mu\varepsilon)^4-2\lambda^2(\mu\varepsilon)^4+(\mu\varepsilon)^6}.$$

The equations  $\alpha_i = \pm 1$  can, although the polynomial is of degree 5, be solved for one variable, see Fig. 2.

## APPENDIX C: JURY COEFFICIENTS FOR UNMODIFIED DIFFERENCE CONTROL

For  $\tau=1$ , the Jury coefficients are

$$\alpha_3 = \varepsilon\mu, \quad \alpha_2 = \varepsilon\mu(\lambda - 1)(1 - (\varepsilon\mu)^2)^{-1}, \quad \alpha_1 = \frac{((\varepsilon\mu)^2 - \lambda)(1 - (\varepsilon\mu)^2 - (\lambda - 1)\varepsilon\mu)}{(1 - (\varepsilon\mu)^2)^2 - (\varepsilon\mu)^2(\lambda - 1)^2}.$$

For  $\tau=2$ , the Jury coefficients are

$$\alpha_4 = \varepsilon\mu, \quad \alpha_3 = \frac{(-1 + \lambda)\varepsilon\mu}{1 - (\varepsilon\mu)^2}, \quad \alpha_2 = \frac{(1 - \lambda)\varepsilon\mu(\lambda - (\varepsilon\mu)^2)}{-1 + 3(\varepsilon\mu)^2 - 2\lambda(\varepsilon\mu)^2 + \lambda^2(\varepsilon\mu)^2 - (\varepsilon\mu)^4},$$

$$\alpha_1 = \frac{(1 + \varepsilon\mu)(-\lambda + (\varepsilon\mu)^2)}{1 + \varepsilon\mu - \lambda\varepsilon\mu + \lambda^2\varepsilon\mu - 2(\varepsilon\mu)^2 + \lambda(\varepsilon\mu)^2 - (\varepsilon\mu)^3}.$$

For  $\tau=3$ , the Jury coefficients are

$$\alpha_5 = \varepsilon\mu, \quad \alpha_4 = \frac{\varepsilon\mu - \lambda\varepsilon\mu}{-1 + (\varepsilon\mu)^2}, \quad \alpha_3 = \frac{(1 - \lambda)\varepsilon\mu(\lambda - (\varepsilon\mu)^2)}{-1 + 3(\varepsilon\mu)^2 - 2\lambda(\varepsilon\mu)^2 + \lambda^2(\varepsilon\mu)^2 - (\varepsilon\mu)^4},$$

$$\alpha_2 = \frac{(1 - \lambda)\varepsilon\mu(\lambda - (\varepsilon\mu)^2)^2}{-1 + 5(\varepsilon\mu)^2 - 4\lambda(\varepsilon\mu)^2 + 3\lambda^2(\varepsilon\mu)^2 - 2\lambda^3(\varepsilon\mu)^2 + \lambda^4(\varepsilon\mu)^2 - 3\lambda^2(\varepsilon\mu)^4 + (\varepsilon\mu)^6 - 36\lambda(\varepsilon\mu)^8},$$

$$\alpha_1 = \frac{\lambda - \lambda\varepsilon\mu + \lambda^2\varepsilon\mu - (\varepsilon\mu)^2 - \lambda(\varepsilon\mu)^2 + (\varepsilon\mu)^3 - \lambda(\varepsilon\mu)^3 + (\varepsilon\mu)^4}{-1 + \varepsilon\mu - \lambda\varepsilon\mu + \lambda^2\varepsilon\mu - \lambda^3\varepsilon\mu + 3(\varepsilon\mu)^2 - 2\lambda(\varepsilon\mu)^2 + \lambda^2(\varepsilon\mu)^2 - 2\lambda(\varepsilon\mu)^3 - (\varepsilon\mu)^4}.$$

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